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Performance optimization of an irreversible quantum spin refrigeration cycle

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Abstract

The irreversible model of a quantum refrigeration cycle composed of two adiabatic and two isomagnetic field processes is established. The working substance in the cycle consists of many noninteracting spin-$1/2$ systems. The performance of the cycle is investigated, based on the quantum master equation and semi-group approach. The general expressions of several important performance parameters, such as the coefficient of performance, cooling rate, and power input, are given. It is found that the coefficient of performance of this cycle is close analogues of that of classical Carnot cycle. Some performance characteristics curves between the cooling rate and the maximum "temperature" ratio of the working substances are plotted. Further, at high temperatures the optimal relations of the cooling rate and the maximum cooling rate are analyzed in detail. The results obtained are further generalized and discussed, so that they may be directly used to describe the performance of the quantum refrigerator using spin-J systems as the working substance. Finally, the optimum characteristics of the quantum Carnot and Ericsson refrigeration cycles are derived analogously.

Keywords: Spin systems; Quantum refrigeration cycle; Performance parameters

1. Introduction

In recent years, the optimal analysis on the performance characteristics of thermodynamic cycles has been extended to the regime of quantum cycles. The performance of quantum Carnot, Ericsson and Stirling cycles have been intensively studied [1–7]. Furthermore, finite-time thermodynamic analyses of some quantum cycles have also been made [8–15]. Many novel conclusions have been obtained. Besides, the investigations have also dealt with the performance of quantum Brayton cycle [8,16]. In fact, Brayton cycle is one of very importance cycles in engineering thermodynamics. The investigation relative to Brayton cycles has continuously attracted a good deal of attention [17–19]. It has some distinctive merits which are noteworthy in theory and practices.

In classical thermodynamic cycles there are the Stirling cycle, Ericsson cycle, Brayton cycle, etc., besides the Carnot cycle. The performance of the Carnot cycle is independent of the property of the working substance, while the performance of other cycles are, in general, dependent on the property of the working substance [20,21]. Quantum-mechanical cycle differs from the classical cycle in two respects. In one respect the working substances may be the spin systems, harmonic oscillator systems, ideal quantum gases, and micro-particle systems confined to a potential well [22,23]. On the other hand, the relaxation dynamics is modeled by the semigroup approach. The advantage of quantum cycles is that the use of phenomenological heat transfer laws can be avoided.

In the present paper, an irreversible model of quantum Brayton refrigeration cycle using the spin-$1/2$ systems as the working substance is established which is composed of two adiabatic and two isomagnetic field processes. Firstly, The thermodynamics property of a spin-$1/2$ system is given, based on the quantum master equation and semi-group approach. Time evolution of the heat exchange processes is derived which is different from that described previously [13–15]. The performance characteristics of the quantum Brayton refrigeration cycle are analyzed. Secondly, the important performance parameters such as the coefficient of performance, cooling rate, and power input are optimized. Especially, at high temperatures the optimal analytical relation of the cooling rate and the maximum cool-

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Nomenclature

$\mathbf{B}$	magnetic field	T
E	internal energy	J
$\widehat{H}$	Hamiltonian	J
h	Planck constant	J s
$\mathbf{M}$	magnetic moment	J T^{-1}
P	output power	J s^{-1}
Q	amount of heat	J
Q_c	amount of heat absorbed to the working substance from the hot reservoir	J
Q_h	amount of heat released to the hot reservoir from the working substance	J
R	cooling rate	J s^{-1}
r	ratio of high and low "magnetic fields"	
$\mathbf{S}$	spin angular momentum	
S_1	mean value of the spin angular momentum in one adiabatic process	
S_2	mean value of the spin angular momentum in another adiabatic process	
T	absolute temperature	K
T_c	temperature of cold reservoir	K
T_h	temperature of hot reservoir	K
t	cycle period	s
t_c	time of isomagnetic field process (ω_c)	s
t_h	time of isomagnetic field process (ω_h)	s
W	work per cycle	J

Greek symbols

β	inverse of temperature	J^{-1}
β_c	inverse of temperature of cold reservoir	J^{-1}
β_h	inverse of temperature of hot reservoir	J^{-1}
ε	coefficient of performance	
ε_c	coefficient of performance of the Carnot refrigeration cycle	
μ_B	Bohr magnetron	J T^{-1}
ω	"magnetic field"	J
ω_c	low "magnetic field"	J
ω_h	high "magnetic field"	J
τ	"temperature" ratio of high and low reservoirs	
Γ_c	heat conductivity of cold reservoir	s^{-1}
Γ_h	heat conductivity of hot reservoir	s^{-1}

ing rate are derived in detail. Finally, these results obtained here may be generalized for the spin-J systems. The optimum performance of the quantum Ericsson or Carnot refrigeration cycles may be derived similarly.

2. Quantum refrigeration cycle

Let us consider a quantum spin-1/2 system with a magnetic moment $\mathbf{M}$ placed in a magnetic field $\mathbf{B}$. The magnitude of the magnetic field can change over time, but is not allowed to reach zero. The Hamiltonian of the interaction between the magnetic moment $\mathbf{M}$ in the quantum system and the magnetic field $\mathbf{B}$ is given by [24–26]

$$\widehat{H}(t) = -\widehat{\mathbf{M}}\cdot\mathbf{B} = 2\mu_B\widehat{\mathbf{S}}\cdot B = 2\mu_B B_z(t)\widehat{S}_z \tag{1}$$

where μ_B is the Bohr magnetron, $\mathbf{S}$ is a spin angular momentum, $\hbar = h/(2\pi)$, and h is the Planck constant. Throughout this paper we adopt $\hbar = 1$ and define $\omega(t) = 2\mu_B B_z(t)$ for simplicity. ω is positive since the spin angular momentum and magnetic moment are in opposite directions. One can refer to ω rather than B_z as "the field". Thus, the Hamiltonian of an isolated single spin-1/2 system in the presence of the field $\omega(t)$ may be expressed as

$$\widehat{H}(t) = \omega(t)\widehat{S}_z \tag{2}$$

The internal energy of the spin-1/2 system is of the expectation value of the Hamiltonian, i.e.,

$$E = \langle\widehat{H}\rangle = \omega(t)\langle\widehat{S}_z\rangle = \omega S \tag{3}$$

Based on the statistical mechanics, the expectation value of the spin angular momentum S_z is expressed by the following relation [27–29]

$$S = \langle\widehat{S}_z\rangle = -\frac{1}{2}\,\mathrm{th}(\beta\omega/2) \tag{4}$$

where $-1/2 < S < 0$.

In quantum refrigeration cycles, the spin-1/2 system is not only coupled mechanically to the given "magnetic field" $\omega(t)$, but also coupled thermally to a heat reservoir at temperature T. Based on the semi-group formalism [25], the equation of motion of an operator in the Heisenberg picture is given by the quantum master equation, i.e.,

$$\frac{\mathrm{d}\widehat{\mathbf{X}}}{\mathrm{d}t} = i[\widehat{H},\widehat{\mathbf{X}}] + \frac{\partial\widehat{\mathbf{X}}}{\partial t} + L_D(\widehat{\mathbf{X}}) \tag{5}$$

where

$$L_D(\widehat{\mathbf{X}}) = \sum_\alpha \gamma_\alpha\big(\widehat{\mathbf{V}}_\alpha^+[\widehat{\mathbf{X}},\widehat{\mathbf{V}}_\alpha] + [\widehat{\mathbf{V}}_\alpha^+,\widehat{\mathbf{X}}]\widehat{\mathbf{V}}_\alpha\big) \tag{6}$$

is a dissipation term and originates from a thermal coupling of the spin to a heat reservoir, $\widehat{\mathbf{V}}_\alpha$ and $\widehat{\mathbf{V}}_\alpha^+$ are operators in the Hilbert space of the system and are Hermitian conjugates, and γ_α are phenomenological positive coefficients. Substituting $\widehat{\mathbf{X}}$ in Eq. (5) by $\widehat{H}$ and using Eq. (3), one can obtain the rate of change of the internal energy as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\widehat{H}\rangle = \left\langle\frac{\partial\widehat{H}}{\partial t}\right\rangle + \big\langle L_D(\widehat{H})\big\rangle = \frac{\mathrm{d}\omega}{\mathrm{d}t}S + \omega\frac{\mathrm{d}S}{\mathrm{d}t} \tag{7}$$

Comparing Eq. (7) with the time derivative of the first law of thermodynamics

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}t} + \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{8}$$

one can easily find that the instantaneous power is

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \left\langle\frac{\partial\widehat{H}}{\partial t}\right\rangle = \frac{\mathrm{d}\omega}{\mathrm{d}t}S \tag{9}$$

and the instantaneous heat flow is

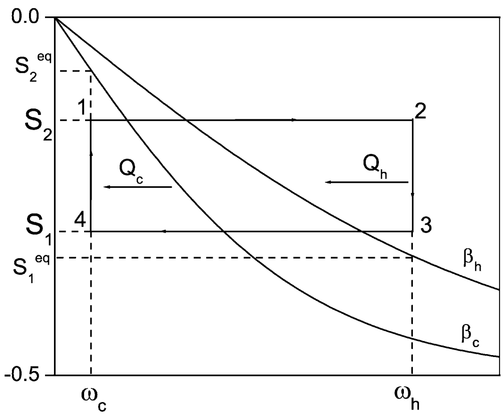

Fig. 1. The S–ω diagram of a spin-1/2 Brayton refrigeration cycle, where the unit of ω is Joule.

$$\frac{dQ}{dt} = \langle L_D(\widehat{H})\rangle = \omega\frac{dS}{dt} \tag{10}$$

It is thus clear that for a spin-1/2 system, Eq. (7) gives the time derivative of the first law of thermodynamics.

Fig. 1 shows a schematic diagram of a quantum refrigeration cycle, which is composed of two adiabatic and two isomagnetic field processes. This cycle is a microscopic analog of the magnetic Brayton refrigeration cycle [30], where the working substance consists of magnetic salts. For the convenience of writing, "temperature" will refer to β rather than T, where $\beta = 1/T$ and T is the absolute temperature in energy units. In adiabatic process $1 \to 2$, no heat exchanges is involved. Increasing ω corresponds to the performance of work by the working substance on the surroundings. The reverse process $3 \to 4$ of adiabatically decreasing ω corresponds to the performance of work by the surroundings on the working substance. In the isomagnetic field process $2 \to 3$, the working substance is coupled to the hot reservoir at constant "temperature" β_h. The amount of heat Q_h is released to the hot reservoir from the working substance. In the isomagnetic field process $4 \to 1$, the working substance is coupled to the cold reservoir at constant "temperature" β_c. The amount of heat Q_c is absorbed to the working substance from the hot reservoir. The "temperatures" of the working substance are different from those of the heat reservoirs. They are, respectively, given by β_1, β_2, β_3, and β_4, and there is a relation, $\beta_4 > \beta_1 \geqslant \beta_c > \beta_h \geqslant \beta_3 > \beta_2$. ω_h and ω_c represent the high and low "magnetic field", respectively.

3. Performance characteristics

Using Eqs. (4) and (10), we can calculate the amounts of heat exchange in two isomagnetic field processes as

$$Q_h = \int_{S_2}^{S_1} \omega\, dS = \omega_h\left[-\frac{1}{2}\,\text{th}(\beta_3\omega_h/2) + \frac{1}{2}\text{th}(\beta_2\omega_h/2)\right] \tag{11}$$

and

$$Q_c = \int_{S_1}^{S_2} \omega\, dS = \omega_c\left[-\frac{1}{2}\,\text{th}(\beta_1\omega_c/2) + \frac{1}{2}\,\text{th}(\beta_4\omega_c/2)\right] \tag{12}$$

where S_1 and S_2 are the mean values of the spin angular momentum in two adiabatic processes respectively and $S_1 < S_2$. During the adiabatic process S remains constant, so there are the following equations:

$$S_1 = -\frac{1}{2}\,\text{th}(\beta_3\omega_h/2) = -\frac{1}{2}\,\text{th}(\beta_4\omega_c/2) \tag{13}$$

and

$$S_2 = -\frac{1}{2}\,\text{th}(\beta_1\omega_c/2) = -\frac{1}{2}\,\text{th}(\beta_2\omega_h/2) \tag{14}$$

Using Eqs. (11)–(12), we obtain the work input per cycle and the coefficient of performance as

$$\begin{aligned} W &= |Q_h + Q_c| \\ &= \omega_h\left[\frac{1}{2}\,\text{th}(\beta_3\omega_h/2) - \frac{1}{2}\,\text{th}(\beta_2\omega_h/2)\right] \\ &\quad + \omega_c\left[\frac{1}{2}\,\text{th}(\beta_1\omega_c/2) - \frac{1}{2}\,\text{th}(\beta_4\omega_c/2)\right] \end{aligned} \tag{15}$$

and

$$\begin{aligned} \varepsilon &= \frac{Q_c}{W} \\ &= \omega_c\left[\frac{1}{2}\,\text{th}(\beta_4\omega_c/2) - \frac{1}{2}\,\text{th}(\beta_1\omega_c/2)\right] \\ &\quad \times \left\{\omega_h\left[\frac{1}{2}\,\text{th}(\beta_3\omega_h/2) - \frac{1}{2}\,\text{th}(\beta_2\omega_h/2)\right]\right. \\ &\quad \left. + \omega_c\left[\frac{1}{2}\,\text{th}(\beta_1\omega_c/2) - \frac{1}{2}\,\text{th}(\beta_4\omega_c/2)\right]\right\}^{-1} \\ &= \frac{\omega_c}{\omega_h - \omega_c} \end{aligned} \tag{16}$$

It is found that the coefficient of performance only depends on high and low "magnetic fields". It is well known that the coefficient of performance of the Carnot refrigeration cycle is $\varepsilon_c = T_c/(T_h - T_c)$. Comparing with these two results, we can find the analogue of the classical thermodynamic result of Carnot cycle as long as temperature T replaced by "magnetic fields" ω.

4. Time evolution of the spin angular momentum and cycle period

In order to calculate the time of the heat-exchange processes, one must solve the equation of motion that determines the time evolution of the spin angular momentum. For a spin system, $\widehat{\mathbf{V}}_\alpha$ are chosen to be the spin creation and annihilation operators: $\widehat{S}_+ = \widehat{S}_x + i\widehat{S}_y$ and $\widehat{S}_- = \widehat{S}_x - i\widehat{S}_y$, and $\widehat{H} = \omega\widehat{S}_z$. Substituting $\widehat{S}_+$, $\widehat{S}_-$, $\widehat{H}$, and $\widehat{\mathbf{X}} = \widehat{S}_z$ into Eq. (5), one can prove that [24]

$$\frac{dS}{dt} = -2(\gamma_+ + \gamma_-)S - (\gamma_- - \gamma_+) \tag{17}$$

If ω is constant, γ_+ and γ_- are also constants, and the solution of Eq. (17) is given by [7]

$$S(t) = S^{\text{eq}} + \left[S(0) - S^{\text{eq}}\right]\exp[-\Gamma t] \tag{18}$$

where $S^{\mathrm{eq}} = -(\gamma_- - \gamma_+)/2(\gamma_- + \gamma_+)$ is the asymptotic value of S and $\Gamma = 2(\gamma_- + \gamma_+)$ is heat conductivity. This asymptotic spin angular momentum must correspond to the value at thermal equilibrium, $S^{\mathrm{eq}} = -\frac{1}{2}\,\mathrm{th}(\beta\omega/2)$. Eq. (18) is a general expression of time evolution for a spin-1/2 system coupling with the heat reservoir and the external magnetic field.

In the isomagnetic field process ω_h, the "temperature" of the working substance changes from β_2 to β_3. Substituting $S(t) = S_1 = -\frac{1}{2}\tanh(\beta_3\omega_h/2)$, $S(0) = S_2 = -\frac{1}{2}\tanh(\beta_2\omega_h/2)$, and $S^{\mathrm{eq}} = S_1^{\mathrm{eq}} = -\frac{1}{2}\tanh(\beta_h\omega_h/2)$ into Eq. (18), one can obtain the time of one isomagnetic field, ω_h, process as

$$t_h = \frac{1}{\Gamma_h}\ln\frac{\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(\beta_2\omega_h/2)}{\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(\beta_3\omega_h/2)} \tag{19}$$

where Γ_h is heat conductivity of hot reservoir. Similarly, substituting $S(t) = S_2 = -\frac{1}{2}\tanh(\beta_1\omega_c/2)$, $S(0) = S_1 = -\frac{1}{2}\tanh(\beta_4\omega_c/2)$, and $S^{\mathrm{eq}} = S_2^{\mathrm{eq}} = -\frac{1}{2}\tanh(\beta_c\omega_c/2)$ into Eq. (18), one can obtain the time of another isomagnetic field, ω_c, process as

$$t_c = \frac{1}{\Gamma_c}\ln\frac{\mathrm{th}(\beta_c\omega_c/2) - \mathrm{th}(\beta_4\omega_c/2)}{\mathrm{th}(\beta_c\omega_c/2) - \mathrm{th}(\beta_1\omega_c/2)} \tag{20}$$

where Γ_c is heat conductivity of cold reservoir. In the two adiabatic processes, since S is a constant of the motion, irrespective of the time dependence of ω, the times spent along the adiabatic processes are negligible. Consequently, the cycle period is given by

$$t = t_h + t_c \tag{21}$$

5. Optimization on performance parameters

The coefficient of performance, cooling rate, and power input are three of the important performance parameters, which are often considered in the optimal design and theoretical analysis of refrigerators. Using Eqs. (12), (15), and (21), one can find that cooling rate and power input may be, respectively, expressed as

$$R = \frac{Q_c}{t} = \omega_c\left[-\frac{1}{2}\,\mathrm{th}(\beta_1\omega_c/2) + \frac{1}{2}\,\mathrm{th}(\beta_4\omega_c/2)\right]\Big/(t_h + t_c) \tag{22}$$

and

$$\begin{aligned} P &= \frac{W}{t} \\ &= \left\{\omega_h\left[\frac{1}{2}\,\mathrm{th}(\beta_3\omega_h/2) - \frac{1}{2}\,\mathrm{th}(\beta_2\omega_h/2)\right]\right. \\ &\quad \left. + \omega_c\left[\frac{1}{2}\,\mathrm{th}(\beta_1\omega_c/2) - \frac{1}{2}\,\mathrm{th}(\beta_4\omega_c/2)\right]\right\}\Big/(t_h + t_c) \end{aligned} \tag{23}$$

Using Eqs. (22)–(23), one can optimize these important performance parameters of the quantum refrigeration cycle. According to Eqs. (13) and (14), Eq. (21) is rewritten as

$$R = \frac{\Gamma\omega_c}{2}\big[\mathrm{th}(y\beta_2\omega_2/2) - \mathrm{th}(\beta_2\omega_h/2)\big][t_h + t_c]^{-1} \tag{24}$$

where

$$\begin{aligned} t_h + t_c &= \ln\frac{[\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(\beta_2\omega_h/2)][\mathrm{th}(\beta_c\omega_c/2) - \mathrm{th}(y\beta_2\omega_c/2)]}{[\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(y\beta_2\omega_c/2)][\mathrm{th}(\beta_c\omega_c/2) - \mathrm{th}(\beta_2\omega_h/2)]} \\ \Gamma &= \Gamma_h = \Gamma_c \quad \text{and} \end{aligned}$$

$y = \beta_4/\beta_2$ is maximum "temperature" ratio of the working substance in the Brayton cycle. Using Eq. (24) and the extremal condition $\partial R/\partial\beta_2 = 0$, we can obtain the following equation

$$\begin{aligned} &(t_h + t_c)(y\omega_c - \omega_h) \\ &\quad\times \frac{[\mathrm{sech}^2(y\beta_2\omega_c/2) - \mathrm{sech}^2(\beta_2\omega_h/2)]}{[\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(\beta_c\omega_c/2)][\mathrm{th}(y\beta_2\omega_c/2) - \mathrm{th}(\beta_2\omega_h/2)]} \\ &\quad - \frac{\omega_h\,\mathrm{sech}^2(\beta_2\omega_h/2)}{[\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(\beta_2\omega_h/2)][\mathrm{th}(\beta_c\omega_c/2) - \mathrm{th}(\beta_2\omega_h/2)]} \\ &\quad + \frac{\omega_c y\,\mathrm{sech}^2(y\beta_2\omega_c/2)}{[\mathrm{th}(\beta_h\omega_h/2) - \mathrm{th}(y\beta_2\omega_c/2)][\mathrm{th}(\beta_c\omega_c/2) - \mathrm{th}(y\beta_2\omega_c/2)]} \\ &= 0 \end{aligned} \tag{25}$$

It gives an optimal relation between y and β_2, but it is too complicate to yield an analytical solution. Based on numerical computational method and Eqs. (24)–(25), we can plot the optimal characteristic curves $R^* \sim y$, as shown in Figs. 2–4, where $R^* = 2R/(\Gamma\omega_c)$ is the dimensionless cooling rate, $r = \omega_h/\omega_c$ is the ratio of high and low "magnetic fields", $\tau = \beta_c/\beta_h$ is the "temperature" ratio of high and low reservoirs and $r > \tau$. The region of maximum "temperature" ratio of the working substance is given

$$r < y < r^2/\tau \tag{26}$$

It is found from these figures that there exist a maximum cooling rate $R_{\max}$ and the corresponding "temperature" ratio of the working substance y_m for a set of given parameters ω_c, β_c, r and τ. For different given parameters, the maximum cooling rate $R_{\max}$ and the corresponding "temperature" ratio of the working substance y_m are different. At given $\omega_c = 6$ J, $\beta_c = 0.5\ \mathrm{J}^{-1}$, $r = 5$, the less τ is, the larger the maximum cooling rate $R_{\max}$ and the corresponding "temperature" ratio of the working substance y_m are, as shown in Fig. 2. At given

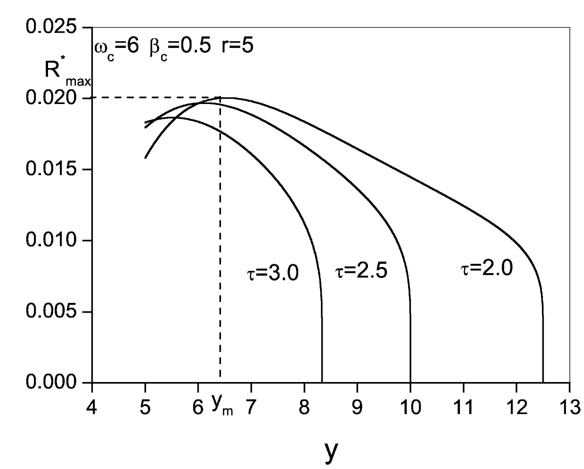

Fig. 2. The dimensional cooling rate $R^* = 2R/(\Gamma\omega_c)$ versus maximum "temperature" ratio of the working substance y for different "temperature" ratio of high and low reservoirs at given parameters $\omega_c = 6$, $\beta_c = 0.5$ and $r = 5$.

$\omega_c = 6$ J, $\beta_c = 0.5$ J^{-1}, $\tau = 2$, the maximum cooling rate $R_{\max}$ are almost unvarying and the corresponding "temperature" ratio of the working substance y_m increase as r increases, as shown in Fig. 3. At given $\beta_c = 0.5$ J^{-1}, $r = 5$, $\tau = 2$, the maximum cooling rate $R_{\max}$ increase, but the corresponding "temperature" ratio of the working substance y_m decrease as ω_c decreases, as shown in Fig. 4. It is found from Figs. 2–4 that among four parameters ω_c, β_c, r and τ the changes of ω_c or τ will largely influence the maximum cooling rate R. But the change of r will almost not affect the maximum cooling rate R. At high temperatures, i.e. $\beta\omega \ll 1$, the results obtained above may be simplified. For example, Eqs. (11)–(16), (19)–(20), and (24)–(25) may be, respectively, simplified as

$$Q_h = \omega_h^2(\beta_2 - \beta_3)/4 \tag{27}$$

$$Q_c = \omega_c^2(\beta_4 - \beta_1)/4 \tag{28}$$

$$\beta_1/\beta_2 = \omega_h/\omega_c = r \tag{29}$$

$$\beta_4/\beta_3 = \omega_h/\omega_c = r \tag{30}$$

$$W = \left[\omega_h^2(\beta_3 - \beta_2) + \omega_c^2(\beta_1 - \beta_4)\right]/4 \tag{31}$$

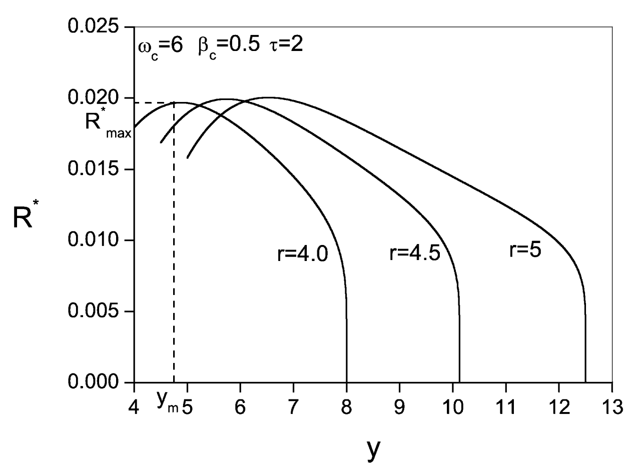

Fig. 3. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y for different the ratio of high and low "magnetic fields" at given parameters $\omega_c = 6$, $\beta_c = 0.5$ and $\tau = 2$.

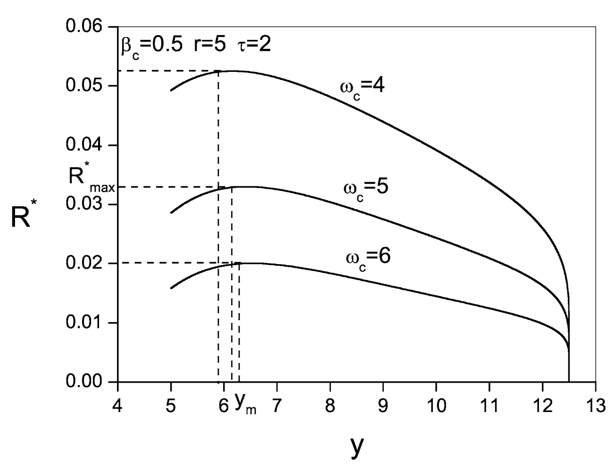

Fig. 4. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y for different low "magnetic fields" at given parameters $\beta_c = 0.5$, $r = 5$ and $\tau = 2$.

$$\varepsilon = \frac{\beta_2}{\beta_1 - \beta_2} = \frac{\beta_3}{\beta_4 - \beta_3} = \frac{1}{r - 1} \tag{32}$$

$$t_h = \frac{1}{\Gamma_h} \ln \frac{\beta_h - \beta_2}{\beta_h - \beta_3} \tag{33}$$

$$t_c = \frac{1}{\Gamma_c} \ln \frac{\beta_c - \beta_4}{\beta_c - \beta_1} \tag{34}$$

$$R = \frac{\Gamma\omega_c}{4}(y\beta_2\omega_c - \beta_2\omega_h) \times \ln \frac{(\beta_h\omega_h - y\beta_2\omega_c)(\beta_c\omega_c - \beta_2\omega_h)}{(\beta_h\omega_h - \beta_2\omega_h)(\beta_c\omega_c - y\beta_2\omega_c)} \tag{35}$$

and

$$\frac{\omega_h}{(\beta_h\omega_h - \beta_2\omega_h)(\beta_c\omega_c - \beta_2\omega_h)} - \frac{\omega_c}{(\beta_h\omega_h - y\beta_2\omega_c)(\beta_c\omega_c - y\beta_2\omega_c)} = 0 \tag{36}$$

It is found that the coefficient of performance only depends on r or the temperature ratio of the original and final states in one adiabatic process. The results obtained here may be compared to those of macroscopic Brayton refrigeration cycle where the working substance consists of magnetic salts [30,31]. It is well known that the coefficient of performance of an ideal gas Brayton refrigeration cycle is $1/(r_P^{1-1/\gamma} - 1)$, where $r_P = P_2/P_1$ is the pressure ratio and γ is the specific heat ratio [32]. It is thus clear that the performance of a Brayton refrigeration cycle depends closely on the properties of the working substance, which is different from that of the Carnot refrigeration cycle. From Eqs. (35)–(36), we can plot the optimal characteristic curves $R^* \sim y$, as shown in Figs. 5–7. It is seen from these figures that the cooling rate monotonically decreases as maximum "temperature" ratio of the working substance increases. At given $\omega_c = 6$ J, $\beta_c = 0.5$ J^{-1}, $r = 5$, the less τ is, the larger the maximum cooling rate $R_{\max}$ is. But the corresponding "temperature" ratio of the working substance y_m is unvarying, as shown in Fig. 5. At given $\omega_c = 6$ J, $\beta_c = 0.5$ J^{-1}, $\tau = 2$, the maximum cooling rate $R_{\max}$ increase and the corresponding "temperature" ratio of the working substance y_m increase as r increases, as shown in Fig. 6. At given $\beta_c = 0.5$ J^{-1}, $r = 5$,

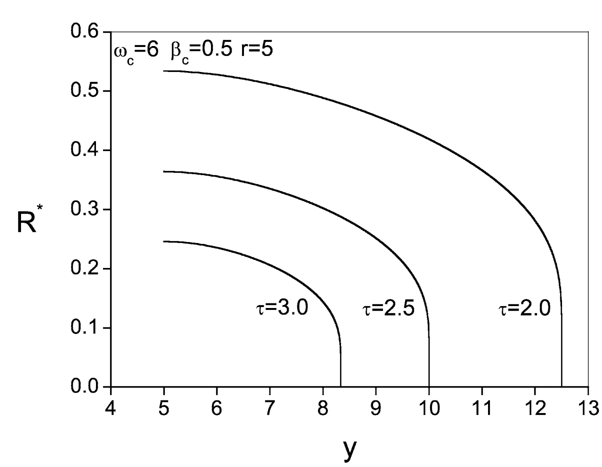

Fig. 5. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y at given parameters $\omega_c = 6$, $\beta_c = 0.5$ and $r = 5$.

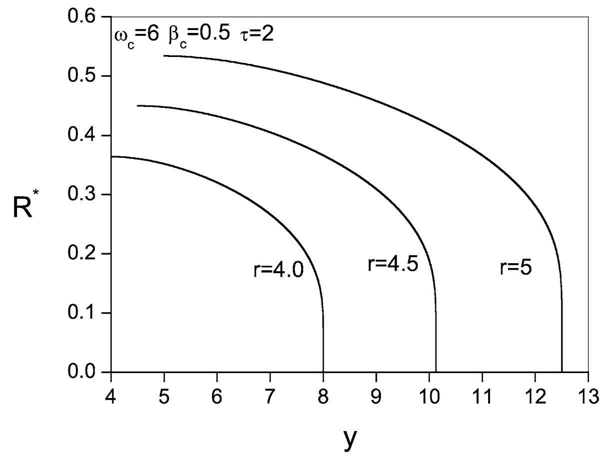

Fig. 6. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y at given parameters $\omega_c = 6$, $\beta_c = 0.5$ and $\tau = 2$.

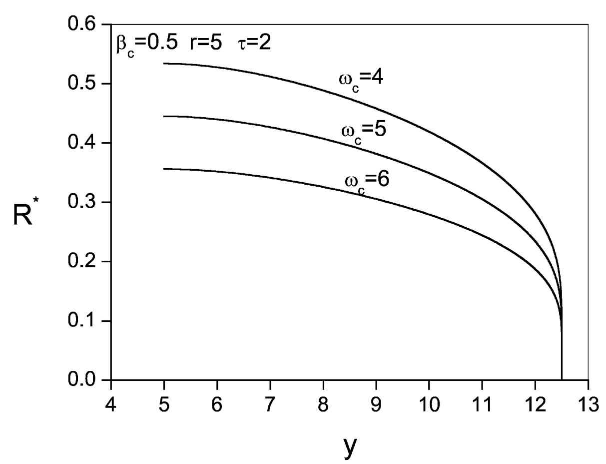

Fig. 7. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y at given parameters $\beta_c = 0.5$, $r = 5$ and $\tau = 2$.

$\tau = 2$, the maximum cooling rate $R_{\max}$ increase, but the corresponding "temperature" ratio of the working substance y_m is unvarying as ω_c decreases, as shown in Fig. 7. These results are different from those in the general case.

6. Discussion and generalizations

(1) If further assumptions are given, i.e. $\beta_3 - \beta_2 \ll \beta_h - \beta_2$, $\beta_4 - \beta_1 \ll \beta_4 - \beta_c$, $\beta_h \geqslant 2\beta_2 - \beta_3$, $\beta_c \leqslant 2\beta_4 - \beta_1$, the times of two isomagnetic field processes may be expressed as

$$t = t_h + t_c = \frac{\beta_3 - \beta_2}{\Gamma_h(\beta_h - \beta_2)} + \frac{\beta_4 - \beta_1}{\Gamma_c(\beta_4 - \beta_c)} \tag{37}$$

The cooling rate R may be simplified as

$$R = \frac{\Gamma \omega_c^2}{4}\left[1/r(\beta_h - \beta_2) + 1/(y\beta_2 - \beta_c)\right]^{-1} \tag{38}$$

Using Eq. (38) and the extremal condition $\partial R/\partial \beta_2 = 0$, we can obtain an optimal relation

$$\beta_2 = \frac{\sqrt{ry}\beta_h + \beta_c}{\sqrt{ry} + y} \tag{39}$$

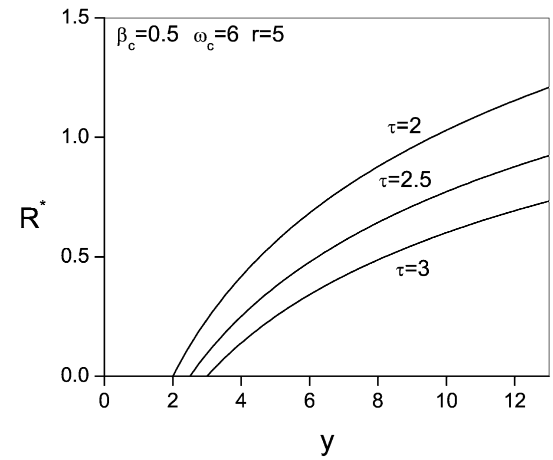

Fig. 8. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y at given parameters $\omega_c = 6$, $\beta_c = 0.5$ and $r = 5$.

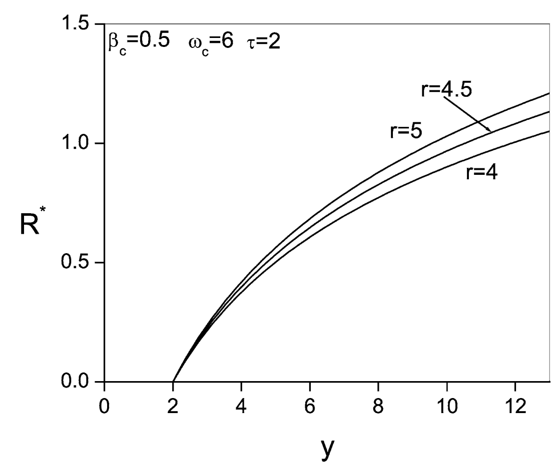

Fig. 9. The dimensionless cooling rate R^* versus maximum "temperature" ratio of the working substance y at given parameters $\omega_c = 6$, $\beta_c = 0.5$ and $\tau = 2$.

Substituting Eq. (39) into Eq. (38), we find that the fundamental optimal relation between the cooling rate and y is given by

$$R = \frac{\Gamma \omega_c^2 r(y\beta_h - \beta_c)}{4(\sqrt{y} + \sqrt{r})^2} \tag{40}$$

Using Eq. (40), we can plot the R^*–y characteristic curves, as shown in Fig. 8 and 9. It is shown that the cooling rate is monotonically increasing function of y. Using Eq. (40), we can obtain

$$\frac{\mathrm{d}R}{\mathrm{d}y} = \frac{\Gamma \omega_c^2 r(\sqrt{r}\beta_h + \beta_c/\sqrt{y})}{4(\sqrt{r} + \sqrt{y})^3} > 0 \tag{41}$$

It is also shown that the cooling rate is monotonically increasing function of y. Substituting $y_{\max} = r^2\beta_h/\beta_c$ into Eq. (40), we can derive the maximum cooling rate,

$$R_{\max} = \frac{\Gamma \omega_c^2 (r^2\beta_h^2 - \beta_c^2)}{4(\sqrt{r\beta_h} + \sqrt{\beta_c})^2} \tag{42}$$

(2) When the work substance is composed of a spin-J system ($J = 1/2, 1, 3/2, 2, \ldots$), the mean value of the spin angular momentum is given by [24–26]

$$S=\langle \widehat{S_z}\rangle=-JB_J(\beta\omega J) \tag{43}$$

where $-J\leqslant S\leqslant J$ and

$$B_J(x)=\left(\frac{2J+1}{2J}\right)\coth\left(\frac{2J+1}{2J}x\right)-\frac{1}{2J}\coth\left(\frac{x}{2J}\right)$$

is the Brillouin function. At high temperatures, Eq. (43) may be simplified as

$$S=-\frac{J(J+1)}{3}\beta\omega \tag{44}$$

Compared with Eq. (4), the heat amount of the two isomagnetic field processes in the cycle may be obtained by multiplying the factor of $4J(J+1)/3$ in Eqs. (27)–(28). On the other hand, using the method in Section 4, one may prove that the time evolution of the spin angular momentum is determined by

$$\frac{\mathrm{d}S}{\mathrm{d}t}=-2a\big\{2S+\beta\omega\big[J(J+1)-M\big]\big\} \tag{45}$$

where $M=\frac{J(J+1)}{3}$. From Eqs. (44) and (45), we can find that the times of the two isomagnetic field processes are the same as Eqs. (33) and (34). Thus, the coefficient of performance of the quantum cycle consisting of the spin-J systems is the same as that of the quantum cycle consisting of the spin-$1/2$ systems, while the cooling rate and power input are $4J(J+1)/3$ times of those of the quantum cycle consisting of the spin-$1/2$ systems, respectively.

(3) When $\beta_3-\beta_2\gg\beta_1-\beta_2$, the two adiabatic processes in the cycle may be replaced by two isothermal processes, the cycle is close to a quantum Ericsson refrigeration cycle [13]. In this case, $\beta_1\approx\beta_2$, and $\beta_3\approx\beta_4$. When $\beta_3-\beta_2\ll\beta_1-\beta_2$, the two isomagnetic field processes in the cycle may be replaced by two isothermal processes, the cycle is close to a quantum Carnot refrigeration cycle [13]. In this case, $\beta_2\approx\beta_3$, and $\beta_1\approx\beta_4$.

(4) The above discussion only refers to a single spin-J system. For the working substance consisting of many noninteracting spin-J systems, the coefficient of performance is still true, while the internal energy, work input, power input, and heat quantity can be obtained as long as the above results are simply multiplied by the total number of spin systems.

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